

3.1~3.9

第三章 Fourier 分析

Fourier 级数
Fourier 变换
通信系统的应用

LTI 系统中

$$\Delta e^{st} = e^{j\omega t} = \cos \omega t + j \sin \omega t \quad \text{复指数信号}$$

卷积

让 $e^{st} * h(t)$ 得到的 $y_{zs}(t)$

$$y_{zs}(t) = \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d\tau$$

$$= e^{st} \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau \quad \text{S 的函数}$$

$$= H(s) \cdot e^{st}$$

连续情况:

$$\Delta S = j\omega, y(t) = H(j\omega) e^{j\omega t}$$

$H(s)$ 特征值

$e^{j\omega t}$ 特征函数

离散情况: z^n , z 是复数, $z = e^s$

$$y(n) = \sum_{k=-\infty}^{+\infty} h(k) \cdot z^{n-k} = z^n \cdot \sum_{k=-\infty}^{+\infty} h(k) \cdot z^{-k} \rightarrow z \text{ 的函数}$$

$$= z^n \cdot H(z) \rightarrow \text{特征值}$$

$$\Delta \text{欧拉公式 } \cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

任意信号都可用无穷多个正弦/余弦函数叠加

Dirichlet 条件

$f(t)$ 满足 ① 周期内有限个间断点

或 ② 极大值极小值有限个

或 ③ 周期内可积

完备正交函数集

矢量正交且唯一 (空间内)

推广到信号

信号的正交: $\int_{t_1}^{t_2} \varphi_1(t) \varphi_2(t) dt = 0$, 则 $\varphi_1(t)$ 与 $\varphi_2(t)$ 正交

$$\textcircled{1} \int_{t_0}^{t_0+T} \cos(m\omega t) \cos(n\omega t) dt$$

$$= \begin{cases} 0 & \text{当 } m \neq n \\ \frac{T}{2} & \text{当 } m = n \neq 0 \\ T & \text{当 } m = n = 0 \end{cases}$$

完备正交函数集

$$\int_{t_1}^{t_2} \varphi_i(t) \varphi_j(t) dt = \begin{cases} 0 & i \neq j \\ \frac{T}{2} & i = j \end{cases}$$

且不存在集合外函数使 $\int_{t_1}^{t_2} \varphi_i(t) \psi(t) dt = 0$

若有相位, 则用三角公式展开

复函数正交

$$\int_{t_1}^{t_2} \varphi_i(t) \varphi_j^*(t) dt = \begin{cases} 0 & i \neq j \\ \frac{T}{2} & i = j \end{cases}$$

*: 共轭

三角形式

一般取上下限 $-\frac{T}{2}, \frac{T}{2}$

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) dt \quad \text{直流分量}$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos(n\omega t) dt \quad \text{偶函数}$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin(n\omega t) dt \quad \text{奇函数}$$

$$\text{证 } f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t)$$

两边乘 $\cos(h\omega t)$, 并取 $-\frac{T}{2}$ 至 $\frac{T}{2}$ 积分

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos(n\omega t) dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} a_0 \cos(n\omega t) dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{m=1}^{\infty} a_m \cos(m\omega t) \cos(n\omega t) dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{m=1}^{\infty} b_m \sin(m\omega t) \cos(n\omega t) dt = 0$$

$$\text{利用正交, 右边} = \int_{-\frac{T}{2}}^{\frac{T}{2}} a_n \cos(n\omega t) \cos(n\omega t) dt = \frac{a_n T}{2}$$

$$\therefore a_n = a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos(n\omega t) dt$$

同理可求 b_n

性质: $a_n = a_{-n}$, $b_n = -b_{-n}$, $-n$ 为数学上分析用

$A > 0, B > 0$

$$\text{同频合成 } A \cos(\omega t) + B \sin(\omega t) = \sqrt{A^2 + B^2} \cos(\omega t - \arctan \frac{B}{A})$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \cos(n\omega t - \arctan \frac{b_n}{a_n})$$

$$= C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega t + \varphi_n)$$

$$C_0 = a_0, C_n = \sqrt{a_n^2 + b_n^2}, \varphi_n = \arg \frac{b_n}{a_n} \quad (\text{见复变})$$

$$= \begin{cases} \arctan \frac{b_n}{a_n} & a_n > 0 \quad \text{一四象限} \\ \pi + \arctan \frac{b_n}{a_n} & \text{二象限} \\ \arctan \frac{b_n}{a_n} - \pi & \text{三象限} \end{cases}$$

△ 指数形式

$$\begin{aligned}
 f(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_1 t) + b_n \sin(n\omega_1 t) \\
 &= a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2} (e^{jn\omega_1 t} + e^{-jn\omega_1 t}) + \frac{b_n}{2j} (e^{jn\omega_1 t} - e^{-jn\omega_1 t}) \\
 &= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - jb_n}{2} e^{jn\omega_1 t} + \frac{a_n + jb_n}{2} e^{-jn\omega_1 t} \right) \\
 \text{令 } F(n\omega_1) &= \frac{a_n - jb_n}{2}, F(-n\omega_1) = \frac{a_n + jb_n}{2} \\
 &= a_0 + \sum_{n=1}^{\infty} F(n\omega_1) e^{jn\omega_1 t} + \sum_{n=-\infty}^{-1} F(n\omega_1) e^{jn\omega_1 t}
 \end{aligned}$$

$n=0$ 时 $F(0) = a_0$

$$\begin{aligned}
 &= \sum_{n=-\infty}^{+\infty} F(n\omega_1) e^{jn\omega_1 t}, \text{ 记 } F(n\omega_1) = F_n \\
 &= \sum_{n=-\infty}^{+\infty} F_n e^{jn\omega_1 t}
 \end{aligned}$$

$$F_n = \frac{a_n - jb_n}{2}, \text{ 求法见前}$$

$$\text{可化简得 } F_n = \frac{1}{T_1} \int_0^{T_1} f(t) e^{-jn\omega_1 t} dt$$

$$\text{同时, } F_n = |F_n| e^{j\varphi_n}$$

$$\text{偶函数 } |F_n| = \sqrt{a_n^2 + b_n^2} = \frac{1}{2} C_n \quad F_n = \frac{a_n}{2} + j \cdot \left(-\frac{b_n}{2}\right)$$

奇函数 $\varphi_n = \dots$ 见上

$$F_n + F_{-n} = a_n$$

$$j(F_n - F_{-n}) = b_n$$

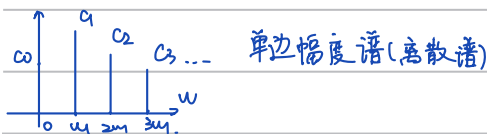
△ 单边幅度谱和相位谱

$$f(t) = C_0 + C_1 \cos(\omega_1 t + \varphi_1) + C_2 \cos(2\omega_1 t + \varphi_2) + \dots + C_n \cos(n\omega_1 t + \varphi_n)$$

↑
基波 (一次谐波)
 ω_1, C_1, φ_1

↑
二次谐波
 $2\omega_1, C_2, \varphi_2$

↑
n次谐波
 $n\omega_1, C_n, \varphi_n$

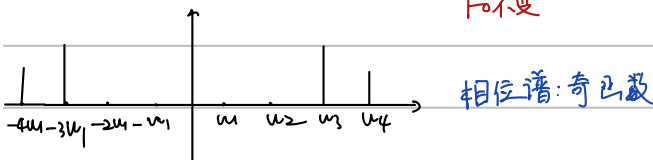


△ 双边幅度谱和相位谱

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n e^{jn\omega_1 t} \quad \text{幅度谱} \quad (\text{能量不变})$$

给出三角, 可先画单边, 再画双边. 同时 $|F_n|$ 变为半

F_0 不变



△ 谱线特性

谱线间距 = $\frac{2\pi}{T}$ 离散谱

$T \rightarrow \infty$ 连续谱, 非周期信号

带宽

△ 傅里叶系数与函数奇偶性

奇函数 $\Rightarrow a_n = 0$

偶函数 $\Rightarrow b_n = 0$

奇谐函数 $f(t) = -f(t \pm \frac{T}{2})$

性质: 只有奇次谐波 1, 3, 5, ..., 偶次谐波均为 0

若一信号去除了直流分量后为奇谐函数, 则只含有奇次谐波与直流

偶谐函数 (选比最小正周期大的周期)

性质: 只有偶次谐波

△ 傅里叶变换

$T \rightarrow \infty, \omega = \frac{2\pi}{T} \rightarrow 0$ 连续谱, 各频率分量幅度 $\rightarrow$ 无穷小

$$\text{频谱(密度)} \quad F(\omega) = \lim_{T \rightarrow \infty} F_n \cdot T_1 = \lim_{\omega \rightarrow 0} F_n \cdot \frac{2\pi}{\omega}$$

(单位频率宽度的频谱值)

$$F_n = \frac{1}{T_1} \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} f(t) \cdot e^{-jn\omega_1 t} dt$$

$$F_n T_1 = \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} f(t) \cdot e^{-jn\omega_1 t} dt \quad \omega_1 = \frac{2\pi}{T_1}, T_1 \rightarrow \infty \Rightarrow \omega_1 \rightarrow 0$$

$$T_1 \rightarrow \infty, F(\omega) = \lim_{T_1 \rightarrow \infty} F_n T_1 = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt \quad (\text{正变换})$$

$$\text{同时, } f(t) = \sum_{n=-\infty}^{+\infty} F_n e^{jn\omega_1 t}$$

$$= \frac{1}{T_1} \sum_{n=-\infty}^{+\infty} F_n \cdot T_1 \cdot e^{jn\omega_1 t}$$

$$\text{令 } T_1 \rightarrow \infty, \text{ 右边} = \frac{2\pi}{T_1} \cdot \frac{1}{2\pi} \cdot \sum_{n=-\infty}^{+\infty} F(\omega) e^{j\omega t}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) \cdot e^{j\omega t} d\omega \quad (\text{逆变换})$$

符号 $\mathcal{F}$ 正变换 $\mathcal{F}[f(t)]$
 $\mathcal{F}^{-1}$ 逆变换 $\mathcal{F}^{-1}[F(\omega)]$ → 复函数.

$$F(\omega) = |F(\omega)| \cdot e^{j\varphi(\omega)} = P(\omega) + jX(\omega)$$

频谱图: 用密度函数表示, 连续谱

若 $F(\omega)$ 为实函数, 则可画在一张图上

$$F(0) = \int_{-\infty}^{+\infty} f(t) dt \Rightarrow f(t) \text{ 的面积} = F(0)$$

$$f(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) d\omega \Rightarrow F(\omega) \text{ 的面积} = 2\pi f(0)$$

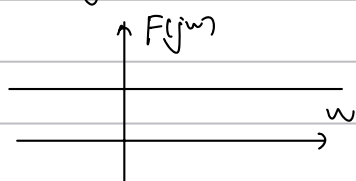
傅里叶变换的条件

$$\int_{-\infty}^{+\infty} |f(t)| dt < \infty \quad \text{绝对可积}$$

△ 常见信号的傅里叶变换

① $\delta(t)$ $\mathcal{F}[\delta(t)] = \int_{-\infty}^{+\infty} \delta(t) \cdot e^{-j\omega t} dt = \int_{-\infty}^{+\infty} \delta(t) dt = 1$

频谱



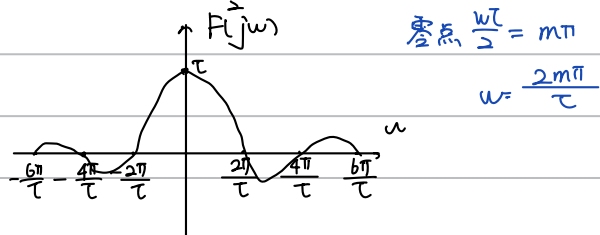
② $\delta'(t)$

$$\mathcal{F}[\delta'(t)] = \int_{-\infty}^{+\infty} \delta'(t) \cdot e^{-j\omega t} dt = -(-j\omega) \cdot 1 = j\omega$$

③ 门函数 $g_\tau(t)$ 宽度为 τ , 高度为 1

$$\mathcal{F}[g_\tau(t)] = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt = \frac{2}{\omega} \frac{-e^{-j\omega \tau/2} + e^{j\omega \tau/2}}{2j}$$

$$= \frac{2}{\omega} \sin \frac{\omega \tau}{2} = \frac{\sin \frac{\omega \tau}{2}}{\frac{\omega}{2}} \cdot \frac{2}{\omega} \cdot \frac{\omega \tau}{2} = \tau \cdot \text{Sa} \frac{\omega \tau}{2}$$



频带宽度 0 ~ 第一个零点 (用表示)

$$\therefore B = \frac{2\pi}{\tau} \cdot \frac{1}{2\pi} = \frac{1}{\tau}$$

④ 单边指数信号 $e^{-at} \cdot \varepsilon(t)$

$$F(j\omega) = \int_{-\infty}^{+\infty} e^{-at} \cdot \varepsilon(t) \cdot e^{-j\omega t} dt$$

$$= \int_0^{+\infty} e^{-(a+j\omega)t} dt$$

$$= \frac{0-1}{-(a+j\omega)} = \frac{1}{a+j\omega}$$

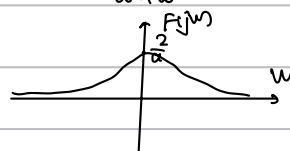
⑤ 双边指数信号 $f(t) = e^{-a|t|}$

$$F(j\omega) = \int_{-\infty}^0 e^{at} \cdot e^{-j\omega t} dt + \int_0^{+\infty} e^{-at} \cdot e^{-j\omega t} dt$$

$$= \frac{1}{a-j\omega} + \frac{-1}{-a-j\omega}$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$= \frac{2a}{a^2 + \omega^2}$$



⑥ 符号函数 $\text{sgn}(t) = \begin{cases} -1 & t < 0 \\ 0 & t = 0 \\ 1 & t > 0 \end{cases}$

不绝对可积, 不能用定义

$$\text{令 } f(t) = \begin{cases} e^{-at} & t > 0 \\ -e^{at} & t < 0 \end{cases}$$

$a \rightarrow 0$ 时, 衰减很慢

$$\therefore \text{sgn}(t) = \lim_{a \rightarrow 0} f(t)$$

$$\mathcal{F}[f(t)] = \int_{-\infty}^0 -e^{at} \cdot e^{-j\omega t} dt + \int_0^{+\infty} e^{-at} \cdot e^{-j\omega t} dt$$

$$= \int_{-\infty}^0 -e^{(a-j\omega)t} dt + \int_0^{+\infty} e^{-(a+j\omega)t} dt$$

$$= \frac{-(-1-0)}{a-j\omega} + \frac{0-1}{-(a+j\omega)}$$

$$= \frac{-1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$= \frac{-a-j\omega + a-j\omega}{a^2 + \omega^2} = \frac{-2j\omega}{a^2 + \omega^2}$$

$$\lim_{a \rightarrow 0} \mathcal{F}[f(t)] = \frac{-2j\omega}{\omega^2} = -\frac{2j}{\omega} = \frac{2}{j\omega} = \frac{2}{\omega} \cdot (-j)$$

$\omega > 0$ 相位 $= -\frac{\pi}{2}$ 幅度 $= \frac{2}{\omega}$

$$= \frac{\pi}{2} \quad = -\frac{2}{\omega}$$

△傅里叶变换的性质

①线性 $f_1(t) \leftrightarrow F_1(j\omega)$, $f_2 \leftrightarrow F_2(j\omega)$

则 $a_1 f_1(t) + a_2 f_2(t) \leftrightarrow a_1 F_1(j\omega) + a_2 F_2(j\omega)$

②对称性

$f(t) \leftrightarrow F(j\omega)$, 则 $F(t) \leftrightarrow 2\pi f(-\omega)$

例 ①常数 C $\delta(t) \leftrightarrow 1$

则 $1 \leftrightarrow 2\pi \delta(-\omega) = 2\pi \delta(\omega)$

由线性 $C \leftrightarrow 2\pi C \delta(\omega)$

②阶跃信号 $\varepsilon(t)$

$\varepsilon(t) = \frac{1}{2} \text{sgn} t + \frac{1}{2}$

$\frac{1}{2} \text{sgn} t + \frac{1}{2} \leftrightarrow \frac{1}{j\omega} + \pi \delta(\omega)$

③ $\text{Sa}(t)$ $g_T(t) \leftrightarrow T \text{Sa}(\frac{\omega T}{2})$

令 $T=2$ $g_2(t) \leftrightarrow 2 \text{Sa}(\omega)$

$2 \text{Sa}(t) \leftrightarrow 2\pi g_2(-\omega) = 2\pi g_2(\omega)$

$\therefore \text{Sa}(t) \leftrightarrow \pi g_2(\omega)$

④ t $\delta'(t) \leftrightarrow j\omega$

$j t \leftrightarrow 2\pi \delta'(-\omega) = -2\pi \delta'(\omega)$

$\therefore t \leftrightarrow -2\pi j \cdot \delta'(\omega)$

⑤ $\frac{1}{t}$ $\text{sgn}(t) \leftrightarrow \frac{2}{j\omega}$

$\frac{2}{j\omega} \leftrightarrow 2\pi \text{sgn}(-\omega) = -2\pi \text{sgn}(\omega)$

$\therefore \frac{1}{t} \leftrightarrow -j\pi \text{sgn}(\omega)$

③尺度变换 $f(t) \leftrightarrow F(\omega)$, 则 $f(at) \leftrightarrow \frac{1}{|a|} F(\frac{\omega}{a})$

(时域扩展, 频域压缩)

$\int_{-\infty}^{+\infty} f(x) \cdot e^{-j\omega x} dx = F(\omega)$

$a=-1$, $f(-t) = F(-\omega)$ 时域反转, 频域反转

例 $\text{Sa}(t) \leftrightarrow \pi g_2(\omega)$

$\text{Sa}(at) \leftrightarrow \frac{\pi}{|a|} g_2(\frac{\omega}{a})$ $\xrightarrow{\text{宽度为 } 2, \omega \text{ 宽度为 } 2a \text{ 的门函数}}$

④时移特性

若 $f(t) \leftrightarrow F(\omega)$, 则 $f(t-t_0) \leftrightarrow F(\omega) e^{-j\omega t_0}$

$f(at-b) \leftrightarrow \frac{1}{|a|} e^{-j\frac{b}{a}\omega} F(\frac{\omega}{a})$

⑤频移特性

$f(t) \leftrightarrow F(\omega)$

则 $f(t) \cdot e^{j\omega_0 t} \leftrightarrow F(\omega - \omega_0)$

例求 $e^{j\omega_0 t} \leftrightarrow ?$

$1 \cdot e^{j\omega_0 t} \leftrightarrow F(\omega - \omega_0)$

$1 \leftrightarrow 2\pi \delta(\omega)$

$\therefore e^{j\omega_0 t} \leftrightarrow 2\pi \delta(\omega - \omega_0)$

求 $\cos \omega_0 t$ $\xrightarrow{2\pi \delta(\omega - \omega_0)}$

$\cos \omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \xrightarrow{2\pi \delta(\omega + \omega_0)}$

$\cos(\omega_0 t + \varphi) = \frac{1}{2} (e^{j\omega_0 t} \cdot e^{j\varphi} + e^{-j\omega_0 t} \cdot e^{-j\varphi})$

$\leftrightarrow \pi [\delta(\omega - \omega_0) \cdot e^{j\varphi} + \delta(\omega + \omega_0) \cdot e^{-j\varphi}]$

求 $\sin \omega_0 t$

$\sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \leftrightarrow \frac{\pi}{j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$

$\sin(\omega_0 t + \varphi) \leftrightarrow \frac{\pi}{j} [\delta(\omega - \omega_0) \cdot e^{j\varphi} - \delta(\omega + \omega_0) \cdot e^{-j\varphi}]$

作用: 频谱搬移

⑥卷积定理

$f_1(t) \leftrightarrow F_1(\omega)$, $f_2(t) \leftrightarrow F_2(\omega)$

则 $f_1(t) * f_2(t) \leftrightarrow F_1(\omega) \cdot F_2(\omega)$

⑦频域卷积定理

$f_1(t) \cdot f_2(t) \leftrightarrow \frac{1}{2\pi} F_1(\omega) * F_2(\omega)$

信号在时域和频域上不可能都是有限长的

证: 有限长信号 $f(t) = m(t) \cdot g(\frac{t}{a})$ $\xrightarrow{\text{门函数}}$

频域上 $m(\omega) * \text{Sa}(\frac{\omega}{a})$ 无限长信号

⑧ 时域微分

$$\text{若 } f(t) \leftrightarrow F(\omega)$$

$$\text{则 } \frac{df(t)}{dt} \leftrightarrow j\omega F(\omega)$$

$$\frac{d^n f(t)}{dt^n} \leftrightarrow (j\omega)^n F(\omega)$$

使用前提: $f(t)$ 中无直流分量

⑨ 时域积分

$$f(t) \leftrightarrow F(\omega)$$

$$\text{则 } \int_{-\infty}^t f(\tau) d\tau \leftrightarrow \frac{F(\omega)}{j\omega} + \pi F(0) \delta(\omega)$$

$$\text{证 } f(t) * \varepsilon(t) = \int_{-\infty}^{\infty} f(\tau) \varepsilon(t-\tau) d\tau = \int_{-\infty}^t f(\tau) d\tau$$

$$\downarrow \quad \quad \downarrow \quad \quad \downarrow$$
$$F(\omega) \cdot \pi \delta(\omega) \frac{1}{j\omega} = \frac{F(\omega)}{j\omega} + \pi F(0) \delta(\omega)$$

⑩ 频域微分 前提: 无直流分量

$$f(t) \leftrightarrow F(\omega)$$

$$(jt) f(t) \leftrightarrow \frac{dF(\omega)}{d\omega}$$

$$(jt)^n f(t) \leftrightarrow \frac{d^n F(\omega)}{d\omega^n}$$

⑪ 频域积分

$$f(t) \leftrightarrow F(\omega)$$

$$\pi f(0) \delta(t) + \frac{f(t)}{-jt} \leftrightarrow \int_{-\infty}^{\infty} F(\eta) d\eta$$

△ 帕塞瓦尔定理

$$\text{① 功率 } \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |F_n|^2$$

$$\text{② 能量 } \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

常用信号的傅里叶变换

时域

频域

$$\star \delta(t)$$

$$1$$

$$\star \delta'(t)$$

$$j\omega$$

$$\varepsilon(t)$$

$$\pi \delta(\omega) + \frac{1}{j\omega}$$

$$t \varepsilon(t)$$

$$j\pi \delta'(\omega) - \frac{1}{\omega^2}$$

$$c$$

$$2\pi c \delta(\omega)$$

$$\text{sgn}(t)$$

$$\frac{2}{j\omega}$$

$$e^{-at} \cdot \varepsilon(t)$$

$$\frac{1}{a+j\omega}$$

$$e^{-a|t|}$$

$$\frac{2a}{a^2+\omega^2}$$

$$e^{-j\omega_0 t}$$

$$2\pi \delta(\omega+\omega_0)$$

$$\cos(\omega_0 t)$$

$$\pi [\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

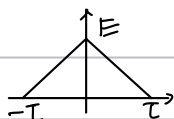
$$\sin(\omega_0 t)$$

$$j\pi [\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$$

$$g_\tau(t)$$

$$\tau \text{Sa}(\frac{\omega\tau}{2})$$

三角脉冲



$$ET \text{Sa}^2(\frac{\omega\tau}{2})$$

$$\text{Sa}(at)$$

$$\frac{\pi}{a} g_{2a}(\omega)$$

$$t$$

$$2\pi j \delta'(\omega)$$

$$\frac{1}{t}$$

$$-j\pi \text{sgn}(\omega)$$

$$\cos(\omega_0 t + \varphi) \quad \pi [\delta(\omega+\omega_0) \cdot e^{-j\varphi} + \delta(\omega-\omega_0) \cdot e^{j\varphi}]$$

$$\sin(\omega_0 t + \varphi) \quad j\pi [\delta(\omega+\omega_0) \cdot e^{-j\varphi} - \delta(\omega-\omega_0) \cdot e^{j\varphi}]$$

$$\text{冲激序列 } \delta_T(t) = \sum_{n=-\infty}^{+\infty} \delta(t-nT_1)$$

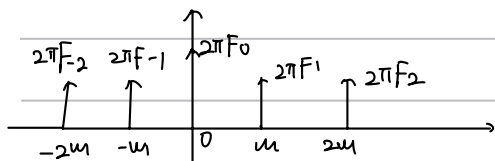
$\updownarrow$

$$\omega_1 \cdot \sum_{n=-\infty}^{+\infty} \delta(\omega-n\omega_1)$$

0 周期信号的傅里叶变换

$$f_T(t) = \sum_{n=-\infty}^{+\infty} F_n \cdot e^{jn\omega_1 t}$$

$$\begin{aligned} \mathcal{F}[f_T(t)] &= \mathcal{F}\left[\sum_{n=-\infty}^{+\infty} F_n \cdot e^{jn\omega_1 t}\right] \\ &= \sum_{n=-\infty}^{+\infty} F_n \cdot \mathcal{F}[e^{jn\omega_1 t}] \\ &= \sum_{n=-\infty}^{+\infty} F_n \cdot 2\pi \delta(\omega - n\omega_1) \\ &= 2\pi \sum_{n=-\infty}^{+\infty} F_n \cdot \delta(\omega - n\omega_1) \end{aligned}$$



强度 = $2\pi \cdot F_n$

$$F_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{-jn\omega_1 t} dt$$

傅里叶变换: $F(\omega) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cdot e^{-j\omega t} dt$

$$\text{则 } F_n = \frac{1}{T} F(\omega_1)$$

$$\begin{aligned} \mathcal{F}[f_T(t)] &= 2\pi \sum_{n=-\infty}^{+\infty} F_n \cdot \delta(\omega - n\omega_1) \\ &= 2\pi \sum_{n=-\infty}^{+\infty} \frac{1}{T} F(\omega_1) \cdot \delta(\omega - n\omega_1) \\ &= \omega_1 \sum_{n=-\infty}^{+\infty} F(\omega_1) \cdot \delta(\omega - n\omega_1) \end{aligned}$$

0 $\delta_T(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$ 的傅里叶变换

$$\delta(t) \leftrightarrow 1$$

$$F(\omega) = 1$$

$$F_n = \frac{1}{T} F(\omega_1) = \frac{1}{T}$$

$$\begin{aligned} \mathcal{F}[\delta_T(t)] &= \frac{2\pi}{T} \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_1) \\ &= \omega_1 \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_1) \end{aligned}$$

任一周期函数均可表示为与 $\delta_T(t)$ 的卷积

$$f_T(t) = f(t) * \delta_T(t)$$

$$\begin{aligned} \therefore \mathcal{F}[f_T(t)] &= F(\omega) \cdot \omega_1 \cdot \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_1) \\ &= \omega_1 \cdot \sum_{n=-\infty}^{+\infty} F(\omega_1) \cdot \delta(\omega - n\omega_1) \end{aligned}$$

△傅里叶分析应用于通信系统

①系统函数

卷积特性: 时域卷积, 频域乘积

$$x(t) * h(t) = y_s(t)$$

↓ 傅里叶变换
系统函数

$$X(j\omega) \cdot H(j\omega) = Y_s(j\omega)$$

$\int_{-\infty}^{+\infty} |h(t)| dt < \infty$ 系统稳定的定义

$$H(j\omega) = \frac{Y_s(j\omega)}{X(j\omega)} = |H(j\omega)| \cdot e^{j\varphi\omega} \quad \text{复函数}$$

$$F(j\omega) = \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega) e^{j\omega t} d\omega$$

$$\delta(t) \longleftrightarrow 1$$

$$1 \longleftrightarrow 2\pi \delta(\omega)$$

$$\varepsilon(t) \longleftrightarrow \pi \delta(\omega) + \frac{1}{j\omega}$$

$$e^{-\sigma t} \varepsilon(t) \longleftrightarrow \frac{1}{\sigma + j\omega}$$

$$g_T(t) \longleftrightarrow \tau \text{Sa}(\frac{\omega\tau}{2})$$

$$\text{sgn}(t) \longleftrightarrow \frac{2}{j\omega}$$

△抽样定理

$$f(t) \cdot \delta(t - t_0) = f(t_0) \cdot \delta(t - t_0) \text{ (取样)}$$

$$\delta_T(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

$f(t) \cdot \delta_T(t)$ 即实现抽样

T : 采样周期

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

△ $f(t) \cdot p(t)$ 的傅里叶变换

$$p(t) \leftrightarrow P(\omega) = \omega_s \cdot \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_s)$$

$$f(t) \leftrightarrow F(\omega)$$

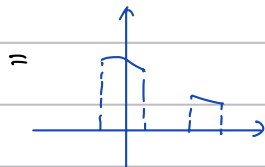
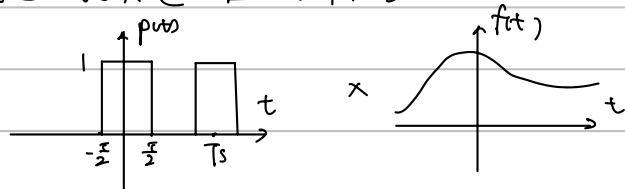
$$f(t) \cdot p(t) \leftrightarrow \frac{\omega_s}{2\pi} \sum_{n=-\infty}^{+\infty} F(\omega - n\omega_s)$$

$$\Delta \text{抽样} \quad f(t) \cdot \sum_{n=-\infty}^{+\infty} \delta(t - nT) = \sum_{n=-\infty}^{+\infty} f(nT) \cdot \delta(t - nT)$$

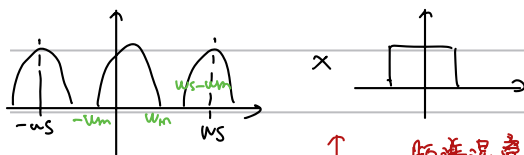
$$\text{搬移} \quad f(t) \rightarrow \sum_{n=-\infty}^{+\infty} \delta(t - nT) = \sum_{n=-\infty}^{+\infty} f(t - nT)$$

△ 现实中无法做到冲激函数

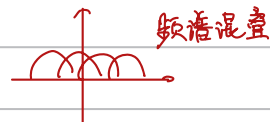
退而求其次: 矩形脉冲信号



△信号的恢复



前提: 波形无交集



条件: $\omega_m < \omega_s - \omega_m$

$\omega_s > 2\omega_m$ 可恢复, ω_m 需是有限值

$\Rightarrow \omega_m$ 需是有限长信号 频带有限

$\omega_s = 2\omega_m$ 时, 有时能恢复, 有时不能恢复

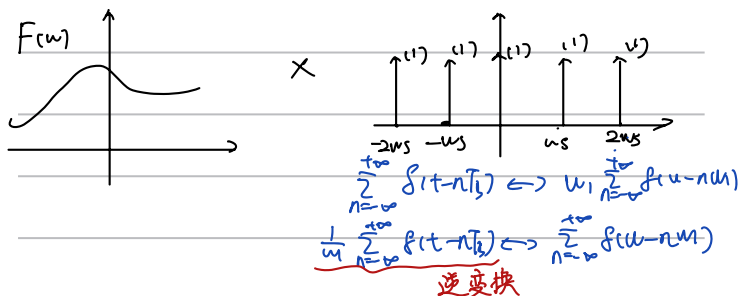
△ 奈奎斯特抽样定理

$\omega_s > 2\omega_m$ ($f_s > 2f_m$, $T_s < \frac{1}{2f_m}$) 时能还原信号

$f_s = 2f_m$ (奈奎斯特频率)

$T_s = \frac{1}{2f_m}$ (奈奎斯特间隔)

△频域抽样定理



$T_s - t_m > t_m$

$T_s > 2t_m$

$f(t) \cdot p(t)$ 的傅里叶变换

$$f(t) \leftrightarrow F(j\omega)$$

$$p(t) \leftrightarrow E\tau\omega_1 \sum_{n=-\infty}^{+\infty} \text{Sa}(\frac{n\omega_1\tau}{2}) \delta(\omega - n\omega_1)$$

$$F(j\omega) * P(j\omega) = E\tau\omega_1 \sum_{n=-\infty}^{+\infty} \text{Sa}(\frac{n\omega_1\tau}{2}) F(\omega - n\omega_1)$$

$$\begin{aligned} f(t) \cdot p(t) &\leftrightarrow \frac{E\tau}{T_1} \sum_{n=-\infty}^{+\infty} \text{Sa}(\frac{n\omega_1\tau}{2}) F(\omega - n\omega_1) \\ &= \frac{E\tau}{T_1} \sum_{n=-\infty}^{+\infty} \text{Sa}(\frac{n\omega_1\tau}{2}) F(\omega - n\omega_1) \end{aligned}$$